

## OVERSHOOTING: MIXING LENGTH YIELDS DIVERGENT RESULTS

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### ABSTRACT

Overshooting (OV) is the signature of the nonlocal nature of convection. To describe the latter, one needs five nonlocal, coupled differential equations to describe turbulent kinetic energies (total  $K$  and in the  $z$ -direction  $K_z$ ), potential energy, convective flux, and rate of dissipation  $\epsilon$ . We show analytically that if  $\epsilon$  is assumed to be given by the local expression,  $\epsilon = K^{3/2}l^{-1}$  (mixing length  $l = \alpha H_p$  or  $l = z/a$ , since the region is small in extent), the remaining differential equations exhibit singularities (divergences) for specific values of  $a$  within the range of values usually employed. No solution can be found. Thus, OV results from such an approach are quite accidental, as they stem from an arbitrary fine tuning of  $a$ .

*Subject headings:* convection — hydrodynamics — stars: evolution — stars: interiors  
 — Sun: evolution — turbulence

### 1. INTRODUCTION

Even when buoyancy forces vanish, eddies retain sufficient residual velocities to “overshoot” into the adjacent, stably stratified (radiative) regions. In massive stars, overshooting (OV) is important because the intruding eddies carry material with bigger mean molecular weight, which ultimately affects the luminosity, since  $L \sim \mu^4 - \mu^{7.5}$ . Early (Prather & Demarque 1974) and recent (Andersen, Nordstrom, & Clauser 1990; Shaller et al. 1992; Nordstrom, Andersen, & Andersen 1997; Kozhurina-Platais et al. 1997) determinations suggest that  $OV = 0.2H_p$ . Stothers & Chin (1991) have suggested an even stronger constraint,  $OV < 0.2H_p$ . In the solar case, one deals primarily with undershooting, which is determined by helioseismological data. Basu, Antia, & Narashima (1994) obtained  $OV = 0.1H_p$ , while Roxburgh & Vorontsov (1994) obtained  $OV = 0.25H_p$ . The most recent analysis, using better data (with lower error) and improvements in the fitting procedure, leads to  $OV = 0.05H_p$  (Basu 1997).

As yet there is no reliable theoretical determination of OV. Numerical simulations (Singh, Roxburgh, & Chan 1995) yield OV values that are too large compared with the new data, while available nonlocal theories (Gough 1977; Xiong 1986) employ a “mixing length,” the implications of which are the subject of this paper.

The problem is as follows. Given the two primary turbulent fields  $u_i$  and  $\theta$ , representing velocity and temperature, respectively (the total velocity and temperature fields are  $v_i = U_i + u_i$  and  $T = \bar{T} + \theta$ ), one constructs five second-order moments:

$$K \equiv \frac{1}{2} \overline{u_i u_i}, \quad K_z \equiv \frac{1}{2} \overline{w^2}, \quad \frac{1}{2} \overline{\theta^2}, \quad J \equiv \overline{w\theta}, \quad \epsilon, \quad (1)$$

representing turbulent kinetic energy, turbulent pressure ( $p_t = 2\rho K_z$ ), temperature variance (potential energy), convective flux ( $F_c = c_p \rho J$ ), and rate of energy dissipation  $\epsilon$ , respectively. To describe turbulent convection, one therefore needs five nonlocal differential equations for the variables in equation (1). These dynamic equations have now been derived using two very different methodologies: a one-point closure (the Reynolds stress approach; Canuto 1992) and a two-point

closure (the stochastic dynamic model; Canuto & Dubovikov 1996, 1997). The equations are the following:

$$\frac{\partial K}{\partial t} + D_f(K) = g\alpha J - \epsilon, \quad (2a)$$

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \overline{\theta^2} \right) + D_f \left( \frac{1}{2} \overline{\theta^2} \right) = \beta J - \tau_\theta^{-1} \overline{\theta^2} + \frac{1}{2} \chi \frac{\partial^2 (\overline{\theta^2})}{\partial z^2}, \quad (2b)$$

$$\frac{\partial J}{\partial t} + D_f(J) = \beta \overline{w^2} + \frac{2}{3} g\alpha \overline{\theta^2} - \tau_{p\theta}^{-1} J + \frac{1}{2} \chi \frac{\partial^2 J}{\partial z^2}, \quad (2c)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{1}{2} \overline{w^2} \right) + D_f \left( \frac{1}{2} \overline{w^2} \right) = & -\frac{5}{2} \tau^{-1} \left( \overline{w^2} - \frac{2}{3} K \right) \\ & + \frac{1}{3} (1 + 2\beta_5) g\alpha J - \frac{1}{3} \epsilon, \end{aligned} \quad (2d)$$

$$\frac{\partial \epsilon}{\partial t} + D_f(\epsilon) = c_1 \frac{\epsilon}{K} g\alpha J - c_2 \frac{\epsilon^2}{K}. \quad (2e)$$

Notice that there is no mixing length of any sort. The timescales  $\tau_\theta$  and  $\tau_{p\theta}$  will be given below. Here,  $c_1 = 1.44$ ,  $c_2 = 1.92$ ,  $\beta_5 = \frac{1}{2}$ ,  $\tau = 2K\epsilon^{-1}$ ; the superadiabatic gradient is

$$\beta = -\frac{\partial T}{\partial z} + g/c_p, \quad (2f)$$

$\chi$  is the radiative conductivity,  $\alpha = T^{-1}$  (for a perfect gas), and  $D_f$  represents the diffusion processes that make the equations nonlocal. In the limit of stationarity and locality,

$$\frac{\partial}{\partial t} \rightarrow 0, \quad (3a)$$

equations (2a)–(2e) become algebraic. The resulting convective flux reproduces the CM model (Canuto & Christensen-Dalsgaard 1997), which is an improvement over the mixing-length theory (MLT) (Stothers & Chin 1997). An OV region is characterized by three features,

$$\beta < 0, \quad J < 0, \quad \text{Pe} < 1, \quad (3b)$$

that is, the temperature is stably stratified, the convective flux is negative, and convection is inefficient ( $\text{Pe} = \omega l \chi^{-1}$  is the Peclet number). If one inserts equation (3b) into equations (2a), (2d), and (2e), one notices that each term on the right-hand side is negative and acts like a sink. There is no local energy source. In order to achieve stationarity, the only sources are the diffusion terms that bring  $K$ ,  $J$ , etc., from where they are created, which is a nonlocal process. In the low-efficiency regime,  $\tau_{p\theta}$  and  $\tau_\theta$  are much smaller than the turbulent timescale  $\tau$ ,

$$\frac{\tau_{p\theta}}{\tau} = (4\pi^2)^{-1} \text{Pe}, \quad \frac{\tau_\theta}{\tau} = 4(7\pi^2)^{-1} \text{Pe}. \quad (4a)$$

As for the diffusion terms, we have in general

$$D_f(K) = -\frac{1}{3} \frac{\partial}{\partial z} \left[ \nu_t \Delta^{-1} \frac{\partial}{\partial z} (K\Delta) \right], \quad (4b)$$

$$D_f(\epsilon) = -\frac{1}{2} \frac{\partial}{\partial z} \left[ \nu_t (1 + \sigma_t^{-1}) \frac{\partial \epsilon}{\partial z} \right]. \quad (4c)$$

$D_f(\frac{1}{2} \bar{w}^2)$  has the same form [see eq. (4b)], with  $K \rightarrow \frac{1}{2} \bar{w}^2$ . The turbulent Prandtl number  $\sigma_t = 0.72$ , and

$$\Delta \equiv K^2 \epsilon^{-4/3}, \quad \nu_t = \frac{2}{25} \frac{K^2}{\epsilon}, \quad \text{Pe} = \frac{4\pi^2 K^2}{125 \epsilon \chi}. \quad (4d)$$

The form of the two remaining diffusion terms depends on whether convection is efficient or inefficient. For the latter case, we have

$$D_f\left(\frac{1}{2} \bar{\theta}^2\right) = -\frac{35}{33} \chi^{-1} \frac{\partial}{\partial z} \left[ \nu_t^2 \Delta^{-2} \frac{\partial}{\partial z} \left( \frac{1}{2} \bar{\theta}^2 \Delta^2 \right) \right], \quad (4e)$$

$$D_f(J) = -\frac{1}{3} \frac{\partial}{\partial z} \left[ \nu_t \Delta^{-1} \frac{\partial}{\partial z} (J\Delta) \right]. \quad (4f)$$

To encompass the alternative form for  $D_f$ , the so-called down-gradient approximation (DGA) (Canuto 1992, eqs. [36a]–[36c]), we shall employ two parameters,  $p$  and  $q$ , where  $p = \frac{1}{3}$  and  $q = 1$  correspond to equations (4b)–(4f), while  $p = 1$  and  $q = 0$  correspond to the DGA. We stress that equations (2a)–(2e) are quite general, while equations (4a), (4d), and (4e) are valid only in the OV region of inefficient convection.

## 2. KEY INGREDIENT: THE DISSIPATION $\epsilon$

The system of equation (2a)–(2e) is closed, and its application to any specific star ought to yield the value of the OV distance. For a complete calculation, one may also want to consider the extension of equations (4a), (4e), and (4f) to the case of efficient convection, where, however, a theory is much less important, since the temperature gradient is very close to being adiabatic.

Suppose that we now take equation (2e) in the local limit,  $\partial/\partial z \rightarrow l^{-1}$ , where  $l$  is a mixing length. Making use of equations (4c) and (4d), we obtain from equation (2e),

$$\epsilon = \frac{K^{3/2}}{l}. \quad (5a)$$

This expression has been used in the nonlocal models of Gough (1977) and Xiong (1986). We may notice that equation (5a) is also the expression for the turbulent kinetic energy

computed by integrating the Kolmogorov spectrum  $E(k) \sim \epsilon^{2/3} k^{-5/3}$  over all wavenumbers from  $k_0 \sim l^{-1}$ . Since  $l$  must be prescribed from outside, we proceed as follows. We call  $R_{1,2}$  the beginning and the end points of the full convective zone (stable and unstable parts). We are interested in the behavior of the differential equations (2a)–(2d) not only in the small stably stratified OV region, but more particularly in the even smaller region near the end point  $R_2$ , where we have to satisfy boundary conditions of the form

$$J(a, R_2) = \epsilon(a, R_2) = 0, \quad (5b)$$

and so forth, for arbitrary  $a$ .

Near  $R_2$ , it is legitimate to expand  $l$  in a Taylor expansion in the variable  $R_2 - r$  and retain the first term of the series. We thus write, in general,

$$l = \frac{h}{a}, \quad h = \text{Min} [|r - R_1|, |r - R_2|]. \quad (5c)$$

In the undershooting region,  $h = r - R_1$ ,  $h' > 0$ ; in the overshooting region,  $h = R_2 - r$ ,  $h' < 0$ . The factor  $a$  cannot be determined a priori, but its value is expected to be near unity.

We shall now show that there exist three values of  $a$ , called  $a_*$ , for which equations (2a)–(2d) exhibit singularities; namely, near the point  $R_2$ , rather than equation (5b) we have

$$J(a_*, r), \quad \epsilon(a_*, r) \rightarrow \infty, \quad (5d)$$

and thus the system of equations (2a)–(2d) has no solutions. We consider equation (2c). Since in the OV region convection is inefficient, and thus  $\chi > \chi_c$ , we can write

$$\tau_{p\theta} \frac{\partial J}{\partial t} + J = \tau_{p\theta} \beta \bar{w}^2 + \frac{1}{2} \tau_{p\theta} r^{-2} \frac{\partial}{\partial r} \left( \chi r^2 \frac{\partial J}{\partial r} \right). \quad (6a)$$

Subtracting equation (2d) from equation (2a) and multiplying by  $\frac{1}{3}$  gives

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \frac{1}{3} K - \frac{1}{2} \bar{w}^2 \right) + \frac{1}{3} D_f(K) - D_f\left(\frac{1}{2} \bar{w}^2\right) \\ &= \frac{5}{2} \tau^{-1} \left( \bar{w}^2 - \frac{2}{3} K \right) - \frac{2}{3} \beta_5 g \alpha J. \end{aligned} \quad (6b)$$

Since the left-hand side is close to zero, we use equation (6b) to eliminate  $\bar{w}^2$ . Equation (6a) then becomes

$$\begin{aligned} & \frac{l^2}{4\pi^2} \frac{\partial J}{\partial t} + (1 - C) \chi J = \frac{5}{12} \left( \frac{l}{\pi} \right)^{8/3} \beta \epsilon^{2/3} \\ & + \frac{1}{8} \left( \frac{l}{\pi} \right)^2 r^{-2} \frac{\partial}{\partial r} \left( r^2 \chi \frac{\partial J}{\partial r} \right), \\ & C \equiv \frac{1}{3} \beta_5 g \alpha \beta \left( \frac{l}{\pi} \right)^{8/3} \epsilon^{-1/3} \chi^{-1}. \end{aligned} \quad (6c)$$

At the same time, equation (2a) becomes, again using equation (5a),

$$\frac{5}{3} \left( \frac{l}{\pi} \right)^{2/3} \epsilon^{-1/3} \frac{\partial \epsilon}{\partial t} + A_0 \epsilon - A_1 \frac{\partial \epsilon}{\partial r} - A_2 \frac{\partial^2 \epsilon}{\partial r^2} = \alpha g J, \quad (6d)$$

where

$$\begin{aligned} A_0 &= 1 - \frac{5}{6} p \pi^{-2} (1 + 2q) \left( l'^2 + 2 \frac{ll'}{r} \right), \\ A_1 &= \frac{5}{6} p \pi^{-2} \left[ 2 \frac{l^2}{r} + (3 + 2q) ll' \right], \\ A_2 &= \frac{5}{6} p \pi^{-2} l'^2. \end{aligned} \quad (6e)$$

Since OV takes place in a region much smaller than the extent  $R_2$ , we can neglect the terms  $l/r \sim h/r$  in the above expressions. Furthermore

$$l = \frac{1}{a} h = \frac{1}{a} (R_2 - r), \quad l' = -\frac{1}{a}. \quad (7a)$$

In the vicinity of  $R_2$ , we write

$$J = -C_1 h^n, \quad \epsilon = C_2 h^m. \quad (7b)$$

Physical solutions are those that correspond to  $C_{1,2} > 0$ . We substitute equation (7b) in equations 6(c) and 6(d) in the stationary regime. Neglecting terms of order  $h/r \ll 1$ , we obtain the solution

$$J = -C_1 h^8, \quad \epsilon = C_2 h^8, \quad (8a)$$

with

$$C_1 = \frac{\delta}{g\alpha} (1 - Q_1)^{-2} (1 - Q_2)^{-3}, \quad (8b)$$

$$C_2 = -\delta (1 - Q_1)^{-3} (1 - Q_2)^{-3}, \quad (8c)$$

where

$$Q_1 \equiv \frac{15p}{2\pi^2} (9 + 2q) a^{-2}, \quad (8d)$$

$$Q_2 \equiv \left[ 7\pi^{-2} a^{-2} + \frac{4}{5} \beta_5 (1 - Q_1) \right] \left[ 1 + \frac{4}{5} \beta_5 (1 - Q_1) \right]^{-1}, \quad (8e)$$

$$\delta^{1/3} \equiv \frac{5}{12} g\alpha \left| \beta \right| \chi^{-1} (\pi a)^{-8/3} > 0. \quad (8f)$$

Since  $C_{1,2}$  must be positive, we conclude that

$$Q_1 > 1, \quad Q_2 < 1. \quad (9a)$$

The values  $p = \frac{1}{3}$  and  $q = 1$  corresponding to equations (4b)–(4e), imply that

$$0.84 < a < 1.67. \quad (9b)$$

The values  $p = 1, q = 0$ , corresponding to the down gradient, imply that

$$0.84 < a < 2.61. \quad (9c)$$

At the points

$$a_* = 0.84, \quad a_* = 1.67, \quad a_* = 2.61, \quad (9d)$$

the functions  $J$  and  $\epsilon$  diverge; yet these values are well within the realm of possible candidates for  $a$ . For example, the value (with  $x$  being the degree of anisotropy of the eddies)

$$a = (1 + x)^{1/2} / 2 = 3^{1/2} / 2 = 0.866 \quad (9e)$$

is often used, and yet it differs by less than 3% from the singularity  $a_* = 0.84$ . This fact has been confirmed numerically (H. M. Antia 1997, private communication).

### 3. CONCLUSIONS

We have shown that a local expression for  $\epsilon$ , even when the other turbulent variables are treated nonlocally, may lead to divergences, unless one fine-tunes the coefficient  $a$ . Two nonlocal models (Gough 1977; Xiong 1986; Balmforth 1992) have employed the local expression of equation (5a). The question then arises as to the reliability of results based on fine-tuning  $a$  when values different by a few percent would allow no solutions to be found, since the equations diverge. The only procedure which is free of these divergences requires that all five variables of equation (1) be described by the full set of nonlocal equations (2a)–(2e). The system of equations is at present being solved for the solar convective zone (Antia et al. 1997).

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### REFERENCES

- Andersen, J., Nordstrom, B. & Clausen, J. V. 1990, *ApJ*, 363, L33  
 Antia, H. M., Basu, S., Canuto, V. M., & Dubovikov, M. S. 1997, in preparation  
 Balmforth, N. J. 1992, *MNRAS*, 255, 603  
 Basu, S. 1997, *MNRAS*, in press  
 Basu, S., Antia, H. M., & Narashima, D. 1994, *MNRAS*, 267, 209  
 Canuto, V. M. 1992, *ApJ*, 392, 218  
 Canuto, V. M., & Christensen-Dalsgaard, J. 1997, *Ann. Rev. Fluid Mech.*, in press  
 Canuto, V. M., & Dubovikov, M. 1996, *Phys. Fluids*, 8(2), 571  
 ———, 1997, *Phys. Rev. Lett.*, 78, 662  
 Gough, D. O. 1977, *ApJ*, 214, 196  
 Kozhurina-Platais, V., Demarque, P., Platais, I., & Orosz, J. A. 1977, *AJ*, 113, 1045  
 Nordstrom, B., Andersen, J., & Andersen, M. I. 1997, *A&A*, in press  
 Prather, M. J., & Demarque, P. 1974, *ApJ*, 193, 109  
 Roxburgh, I. W., & Vorontsov, S. V. 1994, *MNRAS*, 268, 889  
 Shaller, G., Schaerer, D., Meynet, G., & Maeder, A. 1992, *A&AS*, 96, 269  
 Singh, H. P., Roxburgh, I. W., & Chan, K. I. 1995, *A&A*, 295, 703  
 Stothers, R. B., & Chin, Cw. 1991, *ApJ*, 381, L67  
 ———, 1997, *ApJ*, 478, L103  
 Xiong, D. R. 1986, *A&A*, 167, 239